

THE DYNAMICAL BEHAVIOUR OF HOMOGENEOUS SCALAR-FIELD SPACETIMES WITH GENERAL SELF-INTERACTION POTENTIALS

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ABSTRACT. The dynamics of homogeneous Robertson–Walker cosmological models with a self-interacting scalar field source is examined here in full generality, requiring only the scalar field potential to be bounded from below and divergent when the field diverges. In this way we are able to give a unified treatment of all the already studied cases - such as positive potentials which exhibit asymptotically polynomial or exponential behaviors - together with its extension to a much wider set of physically sensible potentials. Since the set includes potentials with negative inferior bound, we are able to give, in particular, the analysis of the asymptotically anti De Sitter states for such cosmologies.

1. INTRODUCTION

Scalar fields have attracted a great deal of attention in cosmology, since the discovery that such a field can act as an "effective" cosmological constant in driving inflation [18]. From this first break-through, which aroused from the simplest conceivable model, i.e. that of a non self-interacting field, the research on scalar field spacetimes has been constantly growing, and quite more general scenarios have been considered, such as scalar fields coupled with perfect fluids, non-minimal couplings, and alternate theories of gravity [1, 2, 3, 4, 5, 6, 7, 8]

A scalar field spacetime can be viewed, like other matter models coupled with gravity such as perfect fluids, as a solution of the Einstein field equations which depends on the choice of the equation of state for the matter. In the case of the scalar field, the role of the equation of state is played by the self-interaction potential V , which is equal to zero in the "standard" inflationary solution. For several reasons, however, it is difficult to believe that this function vanishes. For instance, dimensional reduction of fundamental theories to four dimensions typically gives rise to self-interacting scalar fields with exponential potentials, coupled to four-dimensional gravity. Therefore exponential potentials have been widely considered in the literature (see [27] and references therein) also with the aim of uncovering possible large-scale observable effects ([11, 25, 26, 28]). Of course, however, at the present status of our knowledge the specific functional form of V is quite unsure and, as a consequence, it would be optimal to classify the dynamical behavior of the spacetimes in dependance of all the possible choices of the potential function. Relevant attempts have been made also in this more general direction. Actually, the application of dynamical systems techniques has proven very useful and, due mainly to the works [10, 19, 20, 21, 22, 23, 24], we know the dynamical behavior of scalar field spacetimes for a wide class of *non-negative* potentials; for such potentials therefore, as will be explained below, the contribution of the present paper relies in a simplification and completion of known results.

Non negativity of the potential means that the energy density of the scalar field has a positive lower bound. However, in many issues and especially when string theory comes into play, it becomes relevant to inspect spacetimes in which the potential still

has a lower bound but this bound is negative. From the physical point of view, it should be noted that, although care must be given to the fact that local positivity of energy density might be violated in these spacetimes, under certain somewhat general conditions potentials of this kind generate solutions with positive total energy [17]. In particular, of course, a constant negative potential generates an "equilibrium" state which is just the Anti de Sitter solution (AdS); therefore, a potential with a negative minimum generates a class of spacetimes which have an AdS "equilibrium" (see e.g. [9, 15, 16]). In the present paper we thus study the dynamical behavior of FRW scalar field cosmological models, imposing only very general conditions on the scalar field potential; our hypotheses essentially reduce to ask the potential to be bounded from below and divergent when the field diverges. Within this quite general framework, we give a unified treatment of all the relevant known cases - such as asymptotically polynomial and exponential behaviors - as well as its completion to a much wider set of possible, physically sensible potentials, also in presence of negative (i.e. AdS) minima. It is worth mentioning that our treatment can be applied to the "reverse" problem as well, i.e. to homogeneous scalar field collapse. This issue is treated in companion papers [12, 13, 14].

The paper is organized as follows. First, we formulate the field equations as a first order, regular dynamical system for the scalar field, its time derivative, and the scale factor. Then, we study in full generality the behavior of the trajectories of this system in dependence of the properties of the potential function. Finally, the issue of recollapse and that of stability of AdS spacetimes are addressed. In particular, we show that anti deSitter solutions are the only solutions such that ϕ goes to a critical point of $V(\phi)$ with negative critical value.

2. FORMULATION OF THE PROBLEM

Einstein's field equations for the Robertson–Walker model

$$(1) \quad ds^2 = -dt^2 + a(t)^2 \left[\frac{1}{1 - kr^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right],$$

coupled with the stress–energy tensor

$$(2) \quad 4\pi T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right) g_{\mu\nu},$$

are given by

$$(3) \quad (G_0^0 = 8\pi T_0^0) : \quad -\frac{3(k + \dot{a}^2)}{a^2} = -(\dot{\phi}^2 + 2V(\phi)),$$

$$(4) \quad (G_1^1 = 8\pi T_1^1) : \quad -\frac{(k + \dot{a}^2) + 2a\ddot{a}}{a^2} = (\dot{\phi}^2 - 2V(\phi)).$$

in the unknown functions $a(t), \phi(t)$. These equations imply

$$(5) \quad T_{0;\mu}^\mu = -2\dot{\phi} \left(\ddot{\phi} + V'(\phi) + 3\frac{\dot{a}}{a}\dot{\phi} \right) = 0.$$

We transform the above second order system (3)–(4) into a first order system introducing the additional variables $h = \frac{\dot{a}}{a}, v = \dot{\phi}$. We must exclude solutions such that $\phi(t) = \phi_0$ constant on some interval but $V'(\phi_0) \neq 0$ ¹, so that we are sure that (5)

¹these solutions are easily found by integration of (3) that becomes $\dot{a}^2 = \frac{2V(\phi_0)}{3}a^2 - k$.

holds due to the vanishing of the term in brackets. Therefore (ϕ, v, h) solves the regular system

$$\begin{aligned} (6) \quad & \dot{\phi} = v, \\ (7) \quad & \dot{v} = -V'(\phi) - 3hv \\ (8) \quad & \dot{h} = -h^2 - \frac{2}{3}(v^2 - V(\phi)). \end{aligned}$$

To see that the converse also holds true, we define the function

$$(9) \quad W(\phi, v, h) = h^2 - \frac{1}{3}(v^2 + 2V(\phi)),$$

and argue as follows. Given a solution $(\phi(t), v(t), h(t))$ of (6)–(8), then $\dot{W}(t)$, the derivative of $W(t)$ along the solution, satisfies the differential equation

$$(10) \quad \dot{W} = -2hW,$$

that yields $W(t) = W_0 \exp(-2 \int_0^t h(\tau) d\tau)$. Moreover, we set

$$(11) \quad a(t) = a_0 e^{\int_0^t h(\tau) d\tau},$$

where $a_0 = |W_0|^{-1/2}$ if $W_0 \neq 0$, and $a(0) = 1$ if $W_0 = 0$. Then the identity

$$(12) \quad W(t) = -\frac{k}{a(t)^2}$$

holds, where $k = -\text{sgn}(W_0)$, and so $\phi(t)$ and $a(t)$ solve (3)–(5), from which (4) follows.

We stress some features of the above approach: first, the system obtained is regular, with equilibrium points given by $(\phi_*, 0, h_*)$ such that $V'(\phi_*) = 0$, $h_*^2 = \frac{2}{3}V(\phi_*)$ (notice that $W(\phi_*, 0, h_*) = 0$), that physically correspond to deSitter universe. Given a solution $(\phi(t), v(t), h(t))$, then $(\phi(-t), -v(-t), -h(-t))$ is the time reversed solution. Moreover, the system represents solutions of *all* Robertson–Walker cosmologies: indeed, from (6), the set $W^{-1}(0)$ is invariant by the flow, and by local uniqueness this ensures that the sign of W is invariant along the flow; from (12) we deduce that solutions with W positive, null, or negative, represent scalar field cosmologies with $k = -1, 0, 1$ respectively. From the sign of $h(t)$ one can tell whether, at the instant t , the solution is collapsing ($h < 0$) or expanding ($h > 0$).

3. QUALITATIVE ANALYSIS OF THE EXPANDING MODELS

In this section we consider initially expanding spacetimes, i.e. solutions such that $h(0) = h_0 > 0$. We suppose that the potential $V(\phi)$ is a C^2 function, such that $\lim_{|\phi| \rightarrow \infty} V(\phi) = +\infty$. We will suppose that critical points of V are isolated, and they are either minimum points or nondegenerate maximum points. We also impose the weak energy condition to be satisfied at initial time, i.e. $v_0^2 + 2V(\phi_0) > 0$. Of course, this is not sufficient for the w.e.c. to be satisfied along the evolution, since $V(\phi)$ can attain also negative values. It is useful to introduce the energy function

$$\epsilon = v^2 + 2V(\phi),$$

in such a way that (6)–(8) imply

$$(13) \quad \dot{\epsilon} = -3hv^2.$$

If we let the solution evolve, either there exists $T > 0$ such that $h(T) = 0$, or $h(t) > 0$ for all $t < \sup \mathbb{I}$, where \mathbb{I} is the maximal interval of definition of the solution. In the

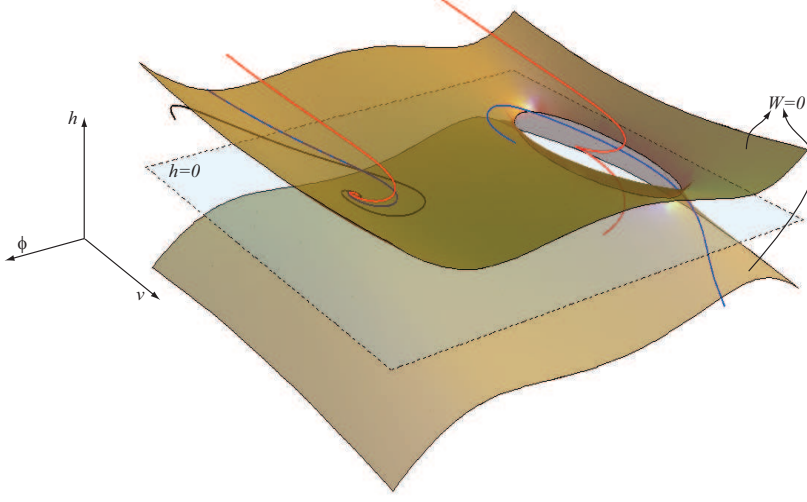


FIGURE 1. Phase portrait sketch when $V(\phi) = 5 - 6x^2 + x^3 + 2x^4$. The behavior when $t > 0$ for some curves with initial data in the open half-space $\{h > 0\}$ is represented. On the left, the situation nearby a local minimum ϕ_* for V , with positive critical value. The three curves on the left side, living in the regions where W is positive, negative and null, respectively, all converge to the equilibrium point $(\phi_*, 0, h_*)$, and therefore the corresponding cosmologies are expanding forever. On the right, the situation nearby a local minimum of V with negative critical value. The two curves, such that W is positive and null, respectively, leave the half space $\{h > 0\}$, therefore passing from expansion to collapsing phase.

latter case, from (10) we deduce that $|W(t)| \leq |W_0|$ and moreover, recalling (13), $\epsilon \leq \epsilon_0 = v_0^2 + 2V(\phi_0)$. Notice also that $\epsilon(t)$ is bounded from below since $V(\phi)$ is, and this implies that both $\phi(t)$ and $v(t)$ are bounded.

Supposing $h(t)$ always positive during the evolution, let us consider separately the cases $W \geq 0$ and $W < 0$. In the first case h decreases, since (8) and (9) imply

$$(14) \quad \dot{h} = -W - v^2,$$

and so h also is bounded. On the other side, $W < 0$ implies $h^2 < \frac{\epsilon}{3}$ and so h is bounded again. In any case the solution lives in a compact set, and this means that $\sup \mathbb{I} = +\infty$, and it is easily proven that it converges to an equilibrium point $(\phi_*, 0, h_*)$ of the system.

Translated into the original formulation, we deduce that if an originally expanding solution does not recollapse, then it is regular for all times, the scalar ϕ converges to a critical point ϕ_* of the potential V , with nonnegative critical value, and $\dot{\phi} \rightarrow 0$.

An example is sketched in Figure 1, when the phase portrait for the system (6)–(8) is sketched, for a potential that has two local minima, one with positive critical value, the other one with negative critical value. In the first case, every solution

starting nearby the critical point $(\phi_*, 0, \sqrt{2V(\phi_*)/3})$ approaches it, regardless of the curvature ($k = -1, 0$, or 1). In the second case, the negative minimum determines an intersection between the two branches of the set $W = 0$, that acts as a “tunnel” allowing solution such that $W \geq 0$ and starting with $h > 0$ to recollapse.

The recollapse problem is an interesting feature of these cosmological models that deserves to be further investigated (see Section 4). At the moment, we observe that (9) – but a simple inspection of (3) also – implies that solutions with $k = -1$ may recollapse only if they violate the w.e.c., and, by the aforesaid, that occurs only if V can attain negative values. We will turn back on this situation later when we will examine stability of anti de Sitter model.

Now we examine stability of the equilibria $(\phi_*, 0, h_*)$ as expanding solutions, where ϕ_* is a critical point of the functional and $h_* = \sqrt{\frac{2V(\phi_*)}{3}}$. Recall that, by the assumptions made on $V(\phi)$, ϕ_* is either a minimum, or a non degenerate maximum. Let us now briefly review the second situation: here, one can simply linearize the system (6)–(8) to find that eigenvalues are always real, and one of them is always positive, and one always negative, which results in the existence of a 1- or 2-dimensional unstable manifold at the equilibrium point, and then, up to a zero-measure set of initial data, the scalar field do not approach the maximum.

Now, consider the case when ϕ_* is local minimum for V , with non negative critical value. If ϕ_* is a non degenerate critical value, asymptotic stability of the equilibrium is found simply using the linearized flow of (6)–(8), and therefore solutions with initial data near the equilibrium point approach it, regardless the case k under consideration. Here, we want to extend to the case ϕ_* is possibly degenerate.

Let us begin considering the situation $V(\phi_*) \geq 0$, in the open topologies $W \geq 0$. Let us consider first the case $V(\phi_*)$ is *strictly* positive. We call $\bar{V} > V(\phi_*)$ a regular value such that ϕ_* is the only critical point in the connected component of $V^{-1}([-\infty, \bar{V}])$ containing ϕ_* , and let us consider a solution $(\phi(t), v(t), h(t))$ with initial data such that $v_0^2 + 2V(\phi_0) \leq 2\bar{V}$, and let $\bar{W} = W(0) > 0$. We define Ω to be the connected component of the set $\{(\phi, v, h) : W(\phi, v, h) \in [0, \bar{W}], \epsilon(\phi, v) \in [2V(\phi_*), 2\bar{V}]\}$ containing the equilibrium point $(\phi_*, 0, \sqrt{\frac{2V(\phi_*)}{3}})$. It is easy to see, using (10) and (13), that Ω is compact, and positively invariant by the flow. So, let us consider a solution with initial data in Ω . Applying LaSalle’s invariance theorem [29] to the functions W and ϵ , we see that the solution must be such that $hW \rightarrow 0$ and $hv^2 \rightarrow 0$. But h is strictly bounded away from zero in Ω , then both W and v must go to zero, which means that the solution approaches the equilibrium point. Since the critical value is strictly positive, then (11) says that the scale factor $a(t)$ diverges at infinity like $\exp(t\sqrt{\frac{2}{3}V(\phi_*)})$.

If $V(\phi_*) = 0$, the above argument can be easily adapted. In this case the set $\{(\phi, v, h) : W(\phi, v, h) \in [0, \bar{W}], \epsilon(\phi, v) \in [2V(\phi_*), 2\bar{V}]\}$ is connected, and we choose Ω to be its subset characterized by the property $h \geq 0$. The only point in Ω with $h = 0$ is exactly the equilibrium point, and so if $h(t) \rightarrow 0$ (recall h is monotone) the solution is forced to approach the equilibrium. If by contradiction $h(t)$ had a strictly positive limit, we can argue as before to find $v \rightarrow 0$ and $W \rightarrow 0$ and so $h \rightarrow 0$. Moreover, when the minimum is non degenerate, then the couple (ϕ, v) has an oscillatory behavior. This fact was already pointed out by Rendall for the flat case [23], but it actually takes place also when $k = -1$. Indeed, taking $\phi_* = 0$ for sake of simplicity, and using the variable change $\phi = \lambda^{-1/2}r \cos \theta$, $v = r \sin \theta$, where

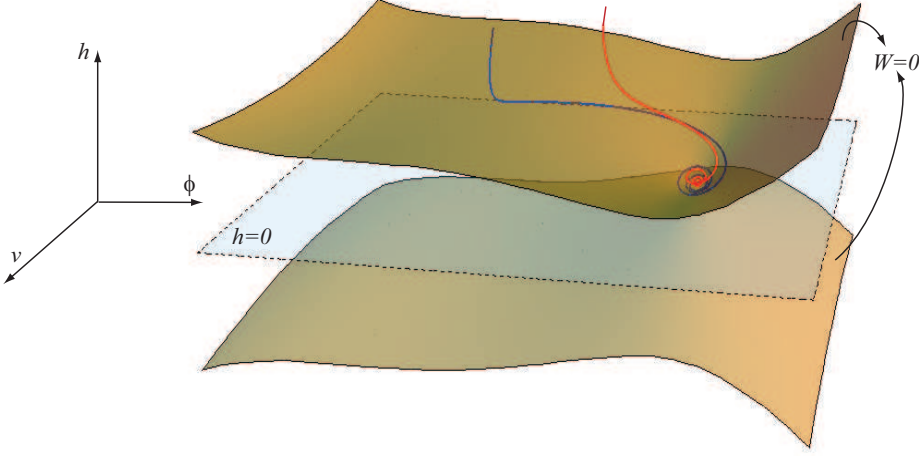


FIGURE 2. Phase portrait sketch when $V(\phi) = x^2 + (2x^3)/3 + x^4/8$. The behavior of two curves with initial data in the half space $\{h > 0\}$, living in the regions where W is positive and null, respectively, is represented. The two curves both approach the equilibrium point $(0, 0, 0)$, where $\phi_* = 0$ is the local minimum of V , with zero critical value. Both curves spin around the equilibrium point for late times.

$2V(\phi) = \lambda\phi^2 + U(\phi)$, with $U(\phi) = o(\phi^2)$, then the equation for θ reads

$$\dot{\theta} = -\sqrt{\lambda} - \frac{3}{2}h \sin 2\theta - \frac{\cos \theta}{2r}U'(\lambda^{-1/2}r \cos \theta),$$

and since both r and h go to zero, we deduce that $\theta(t) = -\sqrt{\lambda}t + o(1/t)$, and this implies the oscillatory behavior of ϕ (and v). An example of this is sketched in Figure 2

Then, it remains proven that every solution of flat or negatively curved Robertson–Walker cosmology with initial data near the local minimum of the potential expands forever approaching this minimum, and the velocity of the scalar field vanishes at infinity.

If $W < 0$, we must adapt the above argument, since now we do not have a priori information on monotony of h (see (14)). As before, we start from the case $V(\phi_*) > 0$, and take \bar{V} and initial data such that $v_0^2 + 2V(\phi_0) \leq 2\bar{V}$, whereas now \bar{W} will be a negative value to determine in such a way that it acts as a lower bound for W . Supposing that initial data are taken near the equilibrium, then h starts positive; since $W(0) \geq \bar{W}$, (10) and (13) imply that $W \geq \bar{W}$ and $\epsilon(t) \leq 2\bar{V}$. Last relation, in particular, shows that $V(\phi(t)) \leq \bar{V}$ and this means that $\phi(t)$ is forced to stay in the potential well $V(\phi_*) \leq V(\phi(t)) \leq \bar{V}$. Then

$$h^2 \geq \frac{\epsilon}{3} + \bar{W} \geq \frac{2V(\phi_*)}{3} + \bar{W} = h_*^2 + \bar{W},$$

that is

$$h \geq \left(\sqrt{1 + \frac{\bar{W}}{h_*^2}} \right) h_* =: \bar{h}.$$

Then, chosen \bar{W} sufficiently small such that $h_*^2 > -\bar{W}$, we have that $h(t) > 0 \Rightarrow h(t) \geq \bar{h} > 0$. Therefore, we can apply again LaSalle's invariance theorem as in the case $W \geq 0$ and exploit the fact that h is bounded away from zero.

All in all, a result of stability of local minima of $V(\phi)$ with positive critical value holds also in the positively curved Robertson–Walker cosmologies. This result, however, cannot be extended as before to the case $V(\phi_*) = 0$, because the equilibrium is such that $h_* = 0$, and so nearby solutions may recollapse (i.e. h can change sign).

4. RECOLLAPSING SOLUTIONS AND THE INSTABILITY OF ANTI DESITTER MODELS

In this section we give some answer to the recollapsing solutions problem sketched before, specifically in the open topology cosmologies, and address the question of stability of anti de Sitter solutions with respect to perturbations in the Robertson–Walker cosmologies.

Let us consider a solution of (6)–(8) with $h(0) > 0$. Since the sign of W is invariant along the flow, we know that $W(0) \geq 0$ implies, using (8), that $h(t)$ is monotone decreasing. If the potential $V(\phi)$ is nonnegative, we already know from previous section that no recollapse will take place, and the solution will expand for all times. So, let \bar{V} be such that $V^{-1}(]-\infty, \bar{V}])$ has a connected component containing only a critical point ϕ_m which is a local minimum with negative critical value. Taking initial data such that $v_0^2 + 2V(\phi_0) \leq 2\bar{V}$, then ϵ and W will decrease until h remains positive, and the solution is forced to remain in the (compact) set such that $W \in [0, W(0)]$, $\epsilon \leq 2\bar{V}$. But there are no equilibrium points to approach in this set, which means that there must be a time T when $h(T) = 0$, and the solution recollapse (notice that this happens violating w.e.c.).

Among these solutions, we can consider anti deSitter models, which occur when $\phi(t) \equiv \phi_m$, and $v(t) = 0$. Let us call $\mu^2 = -\frac{2}{3}V(\phi_m)$; then (8) takes the form $\dot{h} = -h^2 - \mu^2$ that integrates to give $h(t) = -\mu \tan(\mu(t - c))$, where c is the constant of integration. Then $h(t)$ diverges to $-\infty$ in a finite time $t_s > 0$, and so by (9) $W \rightarrow +\infty$, and by (12), $a(t)$ goes to zero, but this is actually a “false” singularity, that does not correspond to a divergence of stress energy tensor which instead remains finite, since v and ϕ are constant.

In order to investigate stability of these solutions, we choose initial data (ϕ_0, v_0) near $(\phi_m, 0)$, and suppose that $(\phi(t), v(t), h(t))$ is a solution of (6)–(13), defined in an interval \mathbb{I} , such that $\phi(t) \rightarrow \phi_m, v(t) \rightarrow 0$. Then $h(t)$ diverges to $-\infty$ as $t \rightarrow \sup \mathbb{I}$. Indeed, by (8), and recalling $V(\phi_m) < 0$, $h(t)$ is eventually decreasing, but if $h(t)$ admits a finite value it would be bounded, so that $\sup \mathbb{I} = +\infty$ and $\dot{h}(t)$ would tend to a negative value, which cannot happen. This fact implies, again, that $a(t)$ goes to zero, that means, using (11), that

$$\lim_{t \rightarrow \sup \mathbb{I}} \int_0^t h(\tau) d\tau = -\infty.$$

Now let us consider equation (7), and call $f(t)$ the (bounded) function $-V'(\phi(t))$. Then it must be

$$\dot{v}(t) + 3h(t)v(t) = f(t),$$

which can be integrated to give a diverging solution, once we prove that $v_0 \neq 0$. Indeed, one can consider situations where $v_0 = 0$, but if $v(t) = 0$ for all time, (7) would give $V'(\phi(t)) = 0$ for all time, that means $\phi(t) = \phi_m$, obtaining anti deSitter model once again. Then v_0 can be taken nonzero without loss of generality, and this results in a diverging solution $v(t)$, which is in contradiction with the assumption ϕ, v bounded. We conclude that anti deSitter solutions are the only solutions of (6)–(8) such that ϕ goes to a critical point of $V(\phi)$ with negative critical value.

5. DISCUSSION AND CONCLUSIONS

We analysed here the dynamics of homogeneous Robertson–Walker cosmological models with a self-interacting scalar field source within a quite general class of (physically valid) potentials $V(\phi)$, essentially requiring only the function V to be bounded from below and divergent when the field diverges. The analysis has been carried out by casting the problem as a regular dynamical system whose solutions describe the possible Robertson Walker cosmologies together with the scalar field evolution.

The asymptotic behavior of the expanding (cosmological) solutions depends crucially on the sign and the extrema of the function V . First of all, if V is positive, the scalar field always approaches the minimum and the solutions do never re-collapse. This result extends and completes already existing results, for instance for exponential potentials, and holds also for the special case of vanishing V at the minimum if the topology is not closed.

The situation complicates drastically for potentials with a negative lower bound. Indeed, solutions which are close to the negative minimum are repulsed; from the cosmological point of view, the corresponding universes re-collapse. The unique exception is the solutions which "sits" on the negative minimum, which of course is the Anti De Sitter space-time. In this sense, AdS turns out to be unstable within homogeneous scalar fields cosmologies.

The phase space in case of several extrema can be sketched using as an example a potential with a positive maximum and one negative and one positive minimum. Clearly, from the results above, the phase space contains ever expanding solutions approaching the positive minimum, and recollapsing solutions. The two sets are separated by solutions for which the scalar field tends to the maximum of the potential. It can be easily shown however that the initial data leading to such a situation form a two-dimensional surface, and therefore these data, besides being of course unstable, are not generic.

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